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A Necessary and Sufficient Condition of Positive Solutions to the BSZ Transformation Model

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1. Introduction

The Transformation Problem (TP) has been causing much controversy since Karl Marx's *Capital* Vol. I. was first published. In particular, after L. von Bortkiewicz proposed a mathematical model for the transformation in 1907, there have been two worldwide debates in more than 90 years, one of which was between P. A. Samuelson and M. Morishima in the early 1970 s). Even now the issue still remains unsettled.

TP is concerned with the transformation from values to production-prices, and mathematical models are used to establish links between two different systems. The basics of TP are to calculate an unknown production pricing system from a *known* value system, namely, to find deviation parameters of production-prices from values.

The difficulty of TP lies in whether it is possible to satisfy ***"two-invariance"*** conditions : total average profit equals total surplus value and total production-price equals to total value. Regarding these two restrictions, there has been no established adequate result so far.

Z. Zhang (2000) has established a mathematical model (the BSZ Transformation Model), in which the ***"two-invariance"*** condition hold, and is consistent with Karl Marx's original intent. The model was based on the coefficient method proposed by L. von Bortkiewicz (1907), and absorbed methods proposed by F. Seton (1957) and P. A. Samuelson (1957). At the same time, Z Zhang gave a stringent mathematical proof of the existence and uniqueness of positive solution to the model and solved the issue of the *unit* of transformation in a lemma.

Recently, Z. Huan found a gap in the proof of the lemma and gave a counter example. After a long discussion with Z. Zhang, Z. Huan found a new proof for the BSZ Transformation Model by considering equivalent systems. The result is presented in this paper.

2. Symbols and the Model

The following is a brief explanation of the symbols used in this paper :

1. c_i, v_i, m_i and w_i represent, the constant capital, variable capital, surplus value and total value in the i^{th} department respectively. c_i+v_i is the total capital of the i^{th} department, and it is also called cost, represented with h_i ; in other words, $h_i=c_i+v_i$.

2. $e(e=m_i / v_i\,;\, i=1,2,\cdots,n)$ represents surplus value rate.

3. $H_i=C_i+V_i$ represents the cost in the i^{th} department, where C_i, V_i are the constant capital and variable capital in the i^{th} department, respectively, under production pricing system.

4. r is average profit rate, $S_i\,(S_i=rH_i=r\,[C_i+V_i])$ is average profit in the i^{th} department.

5. P_i is the total production-price in the i^{th} department ; obviously, $P_i=H_i+S_i=(1+r)H_i$.

It is well known that a mathematical model must be based on certain postulates as its hypotheses. The transformation mathematical model assumes three premises : (1) surplus value rates of all departments are the same ; (2) technology remains unchanged ; (3) for all kinds of capital, the yearly circulation rate is 1.

Z. Zhang (2000) has established the following transformation model

$$
\begin{aligned}
&(1+r)\left(\sum_{j=1}^{n} c_{ij}x_j + v_i y\right) = w_i x_i \quad (i=1,2,\cdots,n) \\
&\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i \\
&r = \sum_{i=1}^{n} m_i \Big/ \sum_{i=1}^{n} (c_i + v_i)
\end{aligned}
\tag{1}
$$

Z. Zhang (2002, 2004) proved the existence of unique positive[1] solution to model (1) in the lemma below.

Lemma 1. *In the following determinant* , $D<0 (n\geq 2)$

$$
D=\begin{vmatrix}
a_1 & b_{12} & \cdots & b_{1,n-1} & d_1 \\
b_{21} & a_2 & \cdots & b_{2,n-1} & d_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
b_{n-1,1} & b_{n-1,2} & \cdots & a_{n-1} & d_{n-1} \\
c_1 & c_2 & \cdots & c_{n-1} & M
\end{vmatrix}
$$

Where, $a_i>0, d_i<0\,(i=1,2,\cdots,n-1), c_j<0\,(j=1,2,\cdots,n-1),\quad M\leq 0, b_{ij}\leq 0\ \ (i,j=1,2,\cdots,n-1;\ i\neq j)$ 。

Unfortunately, Z. Zhang (2002, 2004) overlooked the diagonal element a_i, while he used mathematic induction to prove the lemma. Z. Huan easily found a counter example for the lemma.

Fortunately, the lemma is just a method to establish the results in model (1). We can use equivalent systems to find a new proof for the BSZ Transformation Model

3. Equivalent Systems

Definition 1 (**Z** system and **H** system) :

(1) We call the system

$$\begin{aligned}&[\overline{\mathbf{W}}-(1+r)\mathbf{C}]\mathbf{X}-(1+r)\mathbf{V}y=\mathbf{0}\\&\mathbf{Wx}=\omega\end{aligned} \tag{2}$$

a **Z** system, where $\overline{\mathbf{W}}=diag\,(w_1,w_2,\cdots,w_n)$, $w_i>0$, $i=1,\cdots,n$; $\mathbf{C}=(c_{ij})_{n\times n}$, $c_{ij}\geq 0$, $i, j=1,\cdots,n$; $\mathbf{V}=(v_i)_{n\times 1}$ $v_i>0$, $i=1,\cdots,n$; $\mathbf{W}=(w_i)_{n\times 1}$ $\omega=\sum_{i=1}^{n}w_i$; $\mathbf{X}=(x_i)_{n\times 1}$; y is scalar; and the following relations are satisfied

$$\sum_{i=1}^{n}w_i>\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}+\sum_{i=1}^{n}v_i \tag{3}$$

and

$$r=\frac{\sum_{i=1}^{n}w_i-\left[\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}+\sum_{i=1}^{n}v_i\right]}{\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}+\sum_{i=1}^{n}v_i} \tag{4}$$

(2) We call the system

$$\begin{aligned}&\mathbf{AX}+\mathbf{B}y=0\\&\mathbf{EX}=1\end{aligned} \tag{5}$$

an **H** system, where $\mathbf{A}=(a_{ij})_{n\times n}$, $\mathbf{B}=(b_i)_{n\times 1}$, $\mathbf{E}=(e_i)_{1\times n}$, $\mathbf{X}=(x_i)_{n\times 1}$, $a_{ij}\leq 0$, $(i\neq j), b_i<0, e_i>0, i, j=1,\cdots,n$, and the following relations are satisfied:

$$\sum_{i=1}^{n}e_i=1 \tag{6}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}+\sum_{i=1}^{n}b_i=0 \tag{7}$$

Z systems and **H** systems are equivalent in the following sense.

Theorem 1. (a) For any given **Z** system, there is an **H** system having the same solutions as the **Z** system ; (b) For any given **H** system, there is a **Z** system having the same solutions as the **H** system.

Proof. (a) Assume that (2) is a given **Z** system. Let us define an **H** system possessing the same solution as (2) as below. Set , $\mathbf{A}=\overline{\mathbf{W}}-(1+r)\mathbf{C}$, $\mathbf{B}=-(1+r)\mathbf{V}$ and $\mathbf{E}=\frac{1}{\omega}\mathbf{W}$.

It is clear that
$a_{ij} \le 0$, $(i \ne j)$, $b_i < 0, e_i > 0, i, j = 1, \cdots, n$ and

$$\sum_{i=1}^{n} e_i = \frac{1}{\omega}\sum_{i=1}^{n} w_i = \frac{1}{\omega}\omega = 1$$

as well as the new system have the same solution as the original one.

The rest is to prove that the new system is an **H** system. The relation (6) is verified above. The relation (7) follows from (4)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} w_i - (1+r)\left[\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \sum_{i=1}^{n} v_i\right]$$

$$= \sum_{i=1}^{n} w_i - \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \sum_{i=1}^{n} v_i}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \sum_{i=1}^{n} v_i\right] = 0$$

Therefore, we obtain an **H** system with the same solution as the given Z system.

(b) Assume (5) is a given H system. We have to define a **Z** system possessing the same solution as (5). The key is to figure out what r and w_i are. They have to satisfy the conditions in the definition of S systems. Since w_i must fulfill the inequality (3), we define

$$\lambda_0 = \max\left\{\frac{a_{ii}}{e_i} : i = 1, \cdots, n\right\}$$

The relation (7) implies that $\lambda_0 > 0$. Let $\lambda \ge \lambda_0$ (it is the candidate in the **Z** system) and $w_i = \lambda e_i, \quad i = 1, \cdots, n$). To obtain relation (3), we define

$$\mu_0 = \max\left\{-\frac{\sum_{j=1}^{n} a_{ij} + b_i}{w_i} : i = 1, \cdots, n\right\}$$

The relation (7) tells us that $\mu_0 \ge 0$. Let$r \ge \mu_0 \ge 0$ and define

$$c_{ii} = \frac{w_i - a_{ii}}{1+r}, \qquad i = 1, \cdots, n,$$

$$c_{ij} = \frac{-a_{ij}}{1+r}, \quad i \ne j,\ i, j = 1, \cdots, n$$

and

$$v_i = \frac{-b_i}{1+r}, \qquad i = 1, \cdots, n$$

then

$$\mathbf{A} = \overline{\mathbf{W}} - (1+r)\mathbf{C}, \mathbf{B} = -(1+r)\mathbf{V}, \mathbf{E} = \frac{1}{\lambda}\mathbf{W}$$

The relation (7) implies (4)

$$r = \frac{\sum_{i=1}^{n} w_i - \left[\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \sum_{i=1}^{n} v_i\right]}{\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} + \sum_{i=1}^{n} v_i}$$

Therefore, we have a **Z** system having the same solutions as the given **H** system.

4. A Necessary and Sufficient Conditon for H System Having Positive Solutions and a Corollary

Theorem 2. An **H** system possesses a positive solution if and only if there is $\alpha = (\alpha_i)_{n\times 1}$ satisfying (1) $\alpha_i > 0 \quad (i = 1,2,\cdots,n)$ and (2) $\beta_i = \mathbf{A}\alpha > 0 \quad (i = 1,2,\cdots,n)$.

Proof.

Necessity of the conditions: Assume the **H** system (5) has a positive solution $\mathbf{X}$ = $(x_1,\ldots,x_n, y)$. Setting $\alpha=(x_1,\ldots,x_n)$, we have conditions (1) and (2) immediately

Sufficiency of the conditions: We have $\alpha = (\alpha_i)_{n\times 1}$ satisfying conditions (1) and (2). Let $\Lambda = diag\{\alpha_1, \alpha_2, \cdots \alpha_n\}$, then

$$\mathbf{D} = \Lambda^{-1}\mathbf{A}\Lambda = (d_{ij}) = (a_{ij}\frac{\alpha_j}{\alpha_i})_{n\times n}$$

is a diagonally dominant matrix. To see that, noting $a_{ij} \le 0$ for i≠ j,

$$\sum_{j=1}^{n} d_{ij} = d_{ii} - \sum_{j\ne i}|d_{ij}| = a_{ii} - \sum_{j\ne i}|d_{ij}|$$

and

$$\sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} a_{ij}\frac{\alpha_j}{\alpha_i} = \frac{1}{\alpha_i}\sum_{j=1}^{n} a_{ij}\alpha_j = \frac{\beta_i}{\alpha_i} > 0$$

Therefore, all eigenvalues of **A** are positive.

By $\sum_{j=1}^{n} d_{ij} = a_{ii} - \sum_{j\ne i}|d_{ij}| > 0$, we also know $a_{ii} > 0$. Let $\mu = \max\{a_{ii}, i = 1,2,\cdots,n\}$ (>0) and $\mathbf{A}_0 = \mu\mathbf{I} - \mathbf{A}$. Then $\mathbf{A}_0$ is a nonnegative matrix, the spectral radius $\rho(\mathbf{A}_0)$ also is an eigenvalue of $\mathbf{A}_0$. Consequently $\mu - \rho(\mathbf{A}_0)$ is an eigenvalue of $\mathbf{A}$. Thereupon we gain $\mu > \rho(\mathbf{A}_0)$. Therefore,

$$\mathbf{A}^{-1} = (\mu - \mathbf{A}_0)^{-1} = \frac{1}{\mu}\sum_{k=0}^{\infty}\frac{1}{\mu^k}\mathbf{A}_0^k$$

Since $\mathbf{A}_0$ is nonnegative, we have that $\mathbf{A}^{-1}$ is nonnegative and the solution of (5)

$$\mathbf{X} = \frac{1}{\mathbf{W}'\mathbf{A}^{-1}\mathbf{B}}\mathbf{A}^{-1}\mathbf{B},\quad y = -\frac{1}{\mathbf{W}'\mathbf{A}^{-1}\mathbf{B}}$$ is positive, recalling that $b_i < 0$ and $w_i > 0$.

Corollary If a **Z** system satisfies $(1+r)c_i = (1+r)\sum_{j=1}^{n} c_{ij} < w_i \ (i = 1,2,\cdots,n)$, it has a unique positive solution.

It is simply that $\alpha = (1,1,\cdots,1)'_{n\times 1}$ satisfies the two conditions of Theorem 2.

It is noticeable that the case in corollary is quite important in economics. Therefore, Theorem 2 shows that the BSZ transformation model possesses a wider application range.

Resources

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